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Effect of gravity on a free–free elastic tank partially filled with incompressible liquid

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Abstract

In this paper, we propose a symmetric variational formulation for the eigenmode computation of a free–free elastic tank partially filled with an incompressible inviscid liquid in the presence of a gravity field. The originality of this model is to take into account the strong coupling between the sloshing of the liquid free surface and the hydroelastic deformations of the tank. We will show that this allows the rigid body modes of the system to be predicted correctly.

1. Introduction

In the aerospace field, the study of free structures containing internal fluids is of prime importance, for instance, to predict the in-flight stability of aircrafts with auxiliary tanks or to control the trajectory of liquid propelled launch vehicles (Abramson, 1966; Moïseyev and Rumyantsev, 1968). Several symmetric formulations have been proposed to compute the linear vibrations of elastic tanks partially filled with incompressible inviscid liquids. Some of them account for the sloshing of the liquid free surface with an approximate model for the sloshing wall effect due to gravity (Tong, 1966; Ohayon and Valid, 1984). However, in many industrial applications, incompressible hydroelastic (added-mass) models are used, neglecting the gravity potential energy of the fluid and therefore the sloshing effect. The decoupling of the two phenomena is valid, unless the first coupled eigenfrequencies of the fluid–structure system are too close to the sloshing eigenfrequencies; this may occur for highly flexible tanks but also for free–free systems (since the first eigenmodes are zero-frequency rigid body modes).

In order to take into account the possible coupling between sloshing modes and incompressible hydroelastic modes without gravity, a precise analysis of the linearized problem has been carried out by Morand and Ohayon in 1995. The so-called *elastogravity* stiffness operator they proposed (Morand and Ohayon, 1995) was studied in detail and numerically validated on several applications (Schotté and Ohayon, 2001).

The present paper is devoted toward extending this modelling to free–free fluid–structure systems, considering the uniform translational acceleration as an apparent gravity. It will be demonstrated that, unlike previous approximate hydroelastic models with gravity, the proposed formulation, which accounts for all the prestress and "follower force" terms, correctly represents the rigid body modes of the fluid–structure system and their coupling with the sloshing of the free surface. Numerical simulations and validations are presented to highlight the advantages of this model.

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Nomenclature

Γ	liquid free surface
Σ^i	fluid–structure interface
Σ^f	external force application surface
Ω^S	structural domain ($\partial\Omega^S = \Sigma^i \cup \Sigma^f$)
Ω^F	fluid domain ($\partial\Omega^F = \Gamma \cup \Sigma^i$)
n	normal vector
τ	linear variation of normal vector: $\tau\, \mathrm{d}\Sigma_0 \simeq n\, \mathrm{d}\Sigma - n_0\, \mathrm{d}\Sigma_0$
$\mathrm{D}u$	partial derivative matrix of u
f^D	dead loads
f^F	follower forces
g	gravity field
ρ^S	density of the structure
ρ^F	density of the fluid
H	elastic coefficient tensor of the structure
z	free surface elevation
i_x, i_y, i_z	Cartesian coordinate system defined with unit vector i_z colinear to g

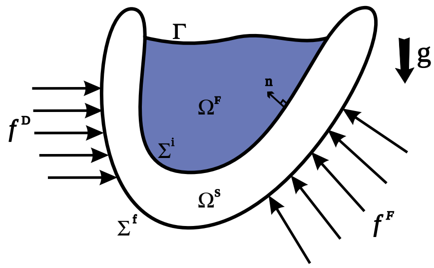

Fig. 1. The model studied.

2. Free–free prestressed fluid–structure system

Herein we are interested in linear vibrations of pulsation ω and we then suppose that all external excitations are harmonic. Fig. 1 shows the system considered here.

2.1. Prestressed initial state and apparent gravity

The presence of a gravity field g and hydrostatic pressure P on the fluid–structure interface introduces a prestress in the initial state of the system (designated by subscript 0) considered as a reference configuration for the vibrational study. In the case of a free–free fluid–structure system, this prestressed initial configuration is a dynamic equilibrium state (frozen at time t) accelerated by the external forces f_0. The initial elastic deformation of the system will be rather described in a reference frame attached to the accelerated system. In this reference frame, gravity is replaced by an apparent gravity that depends on the system transport acceleration, denoted as γ. Since the case of fluid–structure systems in rotation is very specific (Coriolis forces effect etc.) and has been treated in many other studies (Greenspan, 1986), we suppose here that the rigid motion of the system due to the initial external forces is purely translational (this means that the resultant torque of f_0 with respect to the centre of gravity of the system vanishes). In this case, the apparent gravity, denoted as g^{app}, is given by

$$g^{\mathrm{app}} = g - \gamma \quad \text{with } \gamma = \frac{(m^S + m^F)g + \int_{\Sigma_0^f} f_0\, \mathrm{d}\Sigma_0}{m^S + m^F}, \tag{1}$$

thus

$$g^{\mathrm{app}} = -\frac{\int_{\Sigma_0^f} f_0 \, \mathrm{d}\Sigma_0}{m^S + m^F}, \tag{2}$$

where m^S and m^F are the structure and fluid masses.

The knowledge of the apparent gravity (which does not depend on the presence of the gravity field g) allows to position the initial fluid free surface, except in the case of a free fall ($f_0 = 0$) because, when $g^{\mathrm{app}} = 0$, the position of the liquid free surface cannot be defined. Furthermore, in the accelerated reference frame attached to the system, the resultant of all applied loads is zero, since the inertial forces associated to the apparent gravity g^{app} balance the applied loadings f_0. To determine the initial elastic deformations u_0^e and then the prestress σ_0 in the structure, it is however necessary to "support" the unconstrained system by fixing enough degrees of freedom of the structure to make it isostatic (Géradin and Rixen, 1997). The local equations satisfied by u_0^e and σ_0 are:

$$\mathrm{Div}\, \sigma_0(u_0^e) + \rho_0^S g^{\mathrm{app}} = 0 \quad \text{in } \Omega_0^S, \tag{3a}$$

$$\sigma_0 n_0 = f_0 \quad \text{on } \Sigma_0^f, \tag{3b}$$

$$\sigma_0 n_0 = -P_0^{\mathrm{app}} n_0 \quad \text{on } \Sigma_0^i, \tag{3c}$$

where P_0^{app} is the fluid pressure in the reference frame attached to the system.

Let us nevertheless remark that, in the linearized theory proposed here, the geometry of the prestressed initial state is supposed to be very close to its natural geometry (subjected to no external loads).

2.2. Linear hydroelastic vibrations with gravity

The apparent gravity being constant in space, we can apply the hydroelastic modelling with constant gravity developed previously (Schotté and Ohayon, 1999; Schotté, 2001) to describe the linear fluid–structure vibrations in the reference frame attached to the system. We recall here that, if $\mathscr{H}^1(\Omega_0^S)$ is denoted as $\mathfrak{C}_u$, the variational formulation obtained for the deformation U^S of the structure (with respect to the initial configuration) is

$$\exists U^S \in \mathfrak{C}_u, \quad \forall \delta U \in \mathfrak{C}_u, \quad \hat{\mathscr{K}}(U^S, \delta U) - \omega^2 \mathscr{M}(U^S, \delta U) + \omega^2 \mathscr{C}(\varphi, \delta U) = \tilde{f}(\delta U), \tag{4a}$$

where $\hat{\mathscr{K}}$ is the elastogravity operator, $\mathscr{M}$ the inertia of the structure, $\mathscr{C}$ the coupling at the fluid–structure interface, and $\tilde{f}$ the linear form associated with the variation of prescribed forces.

Since the fluid is incompressible and inviscid, its displacements U^F are irrotational at nonzero frequency and then can be represented by a potential φ such as $U^F = \nabla\varphi$ (on a simply connected fluid domain). With $\mathfrak{C}_\varphi^* = \{\varphi \in \mathscr{H}^1(\Omega_0^F) / \int_{\Gamma_0} \varphi \, \mathrm{d}\Gamma_0 = 0\}$, the variational formulation for the liquid is

$$\exists \varphi \in \mathfrak{C}_\varphi^*, \quad \forall \delta\varphi \in \mathfrak{C}_\varphi^*, \quad \mathscr{F}(\varphi, \delta\varphi) - \frac{\omega^2}{|g^{\mathrm{app}}|} \mathscr{S}(\varphi, \delta\varphi) + \mathscr{C}(U^S, \delta\varphi) = 0, \tag{4b}$$

where $\mathscr{F}$ and $\mathscr{S}$ are the bilinear forms associated respectively with the kinetic and sloshing potential energies of the fluid. The detailed expression of all those operators can be found in Schotté and Ohayon (1999).

The symmetric elastogravity operator $\hat{\mathscr{K}}$ [for detailed analysis and properties, see Schotté (2001)] is composed of several bilinear forms: $\hat{\mathscr{K}} = k_E + k_G + k_B + k_\Sigma^1 + k_\Sigma^2$, where

$$k_E(U^S, \delta U) = \int_{\Omega_0^S} \mathrm{Tr}[H\varepsilon(U^S)\varepsilon(\delta U)] \, \mathrm{d}\Omega_0, \tag{5a}$$

$$k_G(U^S, \delta U) = \int_{\Omega_0^S} \mathrm{Tr}[\mathrm{D}U^S \sigma_0 {}^t\mathrm{D}\delta U] \, \mathrm{d}\Omega_0, \tag{5b}$$

$$k_B(U^S, \delta U) = \frac{\rho^F |g^{\mathrm{app}}|}{||\Gamma_0||} \left(\int_{\Sigma_0^i} U^S \cdot n_0 \, \mathrm{d}\Sigma_0 \right) \left(\int_{\Sigma_0^i} \delta U \cdot n_0 \, \mathrm{d}\Sigma_0 \right), \tag{5c}$$

$$k_\Sigma^1(U^S, \delta U) = -\rho^F |g^{\mathrm{app}}| \int_{\Sigma_0^i} (i_z \cdot U^S)(\delta U \cdot n_0) \, \mathrm{d}\Sigma_0, \tag{5d}$$

$$k_\Sigma^2(U^S, \delta U) = -\rho^F |g^{\mathrm{app}}| \int_{\Sigma_0^i} z_0 \tau(U^S) \cdot \delta U \, \mathrm{d}\Sigma_0; \tag{5e}$$

k_E is the symmetric positive bilinear form related to the elastic stiffness of the structure, k_G the symmetric bilinear form representing the geometric stiffness due to the prestress σ_0 in the structure in the initial state, k_B the symmetric positive bilinear form associated with a quasi-static effect of a liquid free surface elevation, and $k_\Sigma = k_\Sigma^1 + k_\Sigma^2$ the symmetric bilinear form related to the "follower force" property of the liquid pressure on the fluid–structure interface.

2.3. Prescribed external follower force effect

We distinguish here two kinds of external forces: the "dead loads" f^D, and the follower forces f^F supposed colinear to the normal vector n. The linearized variation of external loads $\tilde{f}(\delta U)$ appearing in the right-hand side member of Eq. (4a) is then proportional to

$$f\,\mathrm{d}\Sigma - f_0\,\mathrm{d}\Sigma_0 = \tilde{f}^D\,\mathrm{d}\Sigma_0 + \tilde{f}^F n_0\,\mathrm{d}\Sigma_0 + |f_0^F|\tau(U^S)\,\mathrm{d}\Sigma_0, \tag{6}$$

where $\tilde{f}^D = f^D - f_0^D$ and $\tilde{f}^F = |f^F| - |f_0^F|$. The linear variation of the normal vector n, denoted as τ, is a function of U^S. In the case of prescribed follower forces, we then have to add a stiffness term k_F to the elastogravity operator, given by

$$k_F(U^S, \delta U) = -\int_{\Sigma_0^f} |f_0^F|\tau(U^S)\cdot\delta U\,\mathrm{d}\Sigma_0, \tag{7}$$

and to replace the right-hand-side member with

$$\tilde{f}(\delta U) = \int_{\Sigma_0^f} (\tilde{f}^D + \tilde{f}^F n_0)\cdot\delta U\,\mathrm{d}\Sigma_0. \tag{8}$$

The vibrations of the coupled system are free if $\tilde{f}(\delta U) \equiv 0$, which is strictly equivalent to

$$f^D = f_0^D \quad \text{and} \quad |f^F| = |f_0^F|. \tag{9}$$

2.3.1. Symmetry of operator k_F

First of all, let us explicitly define τ in term of U^S using the oriented surface transformation formula:

$$n\,\mathrm{d}\Sigma = \det(F)\,{}^tF^{-1}n_0\,\mathrm{d}\Sigma_0 \quad \text{with } F = Id + \mathrm{D}U^S. \tag{10}$$

Introducing this expression in the definition of $\tau(U^S)$, we find after linearization

$$\tau(U^S) = \mathrm{div}(U^S)n_0 - {}^t(\mathrm{D}U^S)n_0, \tag{11}$$

and then k_F can be written as

$$k_F(U^S, \delta U) = \int_{\Sigma_0^f} |f_0^F|((\mathrm{D}U^S)\delta U - \mathrm{div}(U^S)\delta U)\cdot n_0\,\mathrm{d}\Sigma_0. \tag{12}$$

To study the symmetry of k_F, we evaluate the term $k_F(U,V) - k_F(V,U)$:

$$\begin{aligned} k_F(V,U) - k_F(U,V) &= \int_{\Sigma_0^f} |f_0^F|\mathrm{rot}(V\wedge U)\cdot n_0\,\mathrm{d}\Sigma_0 \\ &= \int_{\partial\Sigma_0^f} |f_0^F|(V\wedge U)\cdot\mathrm{d}l - \int_{\Sigma_0^f} (\mathrm{grad}|f_0^F|\wedge(V\wedge U))\cdot n_0\,\mathrm{d}\Sigma_0. \end{aligned} \tag{13}$$

To enforce the vanishing of this expression, it is sufficient that

(i) on the one hand, $|f_0^F|$ is constant on Σ_0^f ($\mathrm{grad}|f_0^F| \equiv 0$),
(ii) on the other hand, either Σ_0^f is a closed surface ($\partial\Sigma_0^f = 0$) or the displacements of the structure are known on $\partial\Sigma_0^f$ (δU, cinematically admissible, is then null on $\partial\Sigma_0^f$). However, for free–free systems, this second option does not exist.

Both these conditions are verified, for instance, in case of a pressurized tank since $|f_0^F|$ is the pressure of the gas, supposed to be constant in the gas volume and applied on the closed inner surface of the tank.

In the general case, k_F is nonsymmetric, and Hibbitt has proposed using a symmetric formulation instead: $k_F^S = 1/2(k_F + {}^tk_F)$ to preserve the conservative property of the system (Hibbitt, 1979). This approximation is also used by Mohan (1997). Therefore, we suppose in the following that k_F is symmetric.

3. Rigid body modes of a free–free fluid–structure system

The rigid body modes of the free fluid–structure system are solutions of the following equations:

$$\forall(\delta U, \delta\varphi) \in (\mathfrak{C}_u, \mathfrak{C}_\varphi^*), \quad \mathscr{K}(U^S, \delta U) = 0 \quad \text{and} \quad \mathscr{F}(\varphi, \delta\varphi) = -\mathscr{C}(U^S, \delta\varphi) \quad \text{with} \int_{\Gamma_0} \varphi \, \mathrm{d}\Gamma_0 = 0. \tag{14}$$

We are then interested in determining the kernel of the stiffness operator $\mathscr{K} = k_E + k_G + k_B + k_\Sigma^1 + k_\Sigma^2 + k_F$, where these different operators are defined by relations (5). Instead of doing a direct computation, we shall first try to verify whether the six classical rigid body modes U_R (the three translations and the three rotations) belong or not to this space. In this aim, we compute the deformation energy generated by $\mathscr{K}$ for these 6 displacements U_R, defined by $E_{\mathscr{K}}(U_R) = \frac{1}{2}\mathscr{K}(U_R, U_R)$.

3.1. Rigid body translational motions

For a translational motion, $U_R = t$ with $t \in \{i_x, i_y, i_z\}$ and

$$E_{k_E}(t) = 0; \quad E_{k_G}(t) = 0; \quad E_{k_F}(t) = 0; \quad E_{k_\Sigma^2}(t) = 0;$$

$$E_{k_B}(t) = \frac{\rho^F |g^{\mathrm{app}}|}{2\|\Gamma_0\|}\left(t \cdot \int_{\Sigma_0^i} n_0 \, \mathrm{d}\Sigma_0\right)^2; \quad E_{k_\Sigma^1}(r) = -\frac{\rho^F |g^{\mathrm{app}}|}{2}(i_z \cdot t)\left(t \cdot \int_{\Sigma_0^i} n_0 \, \mathrm{d}\Sigma_0\right).$$

By using the relation

$$\int_{\Sigma_0^i} n_0 \, \mathrm{d}\Sigma_0 - \|\Gamma_0\| i_z = \int_{\Sigma_0^i \cup \Gamma_0} n_0 \, \mathrm{d}\Sigma_0 = 0, \tag{15}$$

we show that $E_{\mathscr{K}}(t) = E_{k_E}(t) + E_{k_F}(t) + E_{k_G}(t) + E_{k_B}(t) + E_{k_\Sigma^1}(t) + E_{k_\Sigma^2}(t) = 0$. The translational motions are then rigid body modes for a free fluid–structure system in presence of gravity.

3.2. Rigid body rotational motions

A rotational motion is defined by a rotational vector θ and a centre of rotation O: $U_R = \theta \wedge OM_0$ with $\theta \in \{i_x, i_y, i_z\}$, also denoted as $U_R = \Theta OM_0$, where Θ is the antisymmetric matrix associated to this rotation.

3.2.1. Computation of $E_{k_E}(U_R)$

We have

$$E_{k_E}(U_R) = \frac{1}{2}\int_{\Omega_0^S} \mathrm{Tr}[C\varepsilon(\Theta OM_0)\varepsilon(\Theta OM_0)] \, \mathrm{d}\Omega_0. \tag{16}$$

By using the following relation:

$$\varepsilon_X(\Theta OM_0) = \frac{1}{2}(\mathrm{D}(\Theta OM_0) + {}^t\mathrm{D}(\Theta OM_0)) = \frac{1}{2}(\Theta + {}^t\Theta) = 0, \tag{17}$$

we simplify the previous expression and obtain

$$E_{k_E}(U_R) = 0. \tag{18}$$

3.2.2. Computation of $E_{k_G}(U_R)$

Using the Green formula, we integrate the expression of $E_{k_G}(U_R)$ and obtain

$$\begin{aligned} E_{k_G}(U_R) &= \frac{1}{2}\int_{\Omega_0^S} \mathrm{Tr}[\mathrm{D}U_R\sigma_0{}^t\mathrm{D}U_R] \, \mathrm{d}\Omega_0 \\ &= \frac{1}{2}\int_{\Omega_0^S} (\mathrm{div}({}^t(\mathrm{D}U_R\sigma_0)\delta U_R) - \mathrm{Div}(\mathrm{D}U_R\sigma_0) \cdot U_R) \, \mathrm{d}\Omega_0 \\ &= \frac{1}{2}\int_{\partial\Sigma_0} \mathrm{D}U_R\sigma_0 n_0 \cdot U_R \, \mathrm{d}\Sigma_0 - \frac{1}{2}\int_{\Omega_0^S} \mathrm{Div}(\mathrm{D}U_R\sigma_0) \cdot U_R \, \mathrm{d}\Omega_0. \end{aligned} \tag{19}$$

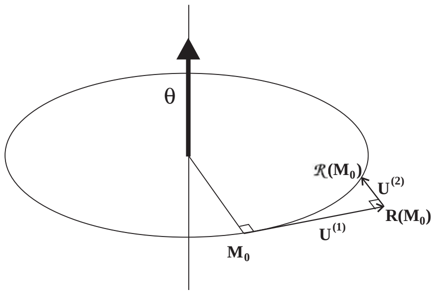

Fig. 2. Infinitesimal and finite rotations.

We then use the local equations (3) and the fact that $\mathrm{D}U_R = \Theta$ to simplify this expression, as follow:

$$E_{k_G}(U_R) = -\frac{1}{2}\int_{\Sigma_0^i} P_0^{\mathrm{app}}(\Theta n_0)\cdot U_R\,\mathrm{d}\Sigma_0 + \frac{1}{2}\int_{\Sigma_0^f}(\Theta f_0)\cdot U_R\,\mathrm{d}\Sigma_0 + \frac{1}{2}\int_{\Omega_0^S}\rho_0^S(\Theta g^{\mathrm{app}})\cdot U_R\,\mathrm{d}\Omega_0. \tag{20}$$

The centre of gravity G_0^S of the structure in its initial configuration is finally introduced to simplify this expression:

$$E_{k_G}(U_R) = \frac{1}{2}\int_{\Sigma_0^i} P_0^{\mathrm{app}} n_0\cdot(\theta\wedge(\theta\wedge OM_0))\,\mathrm{d}\Sigma_0 - \frac{1}{2}\int_{\Sigma_0^f} f_0\cdot(\theta\wedge(\theta\wedge OM_0))\,\mathrm{d}\Sigma_0 - \frac{m^S}{2}g^{\mathrm{app}}\cdot(\theta\wedge(\theta\wedge OG_0^S)). \tag{21}$$

To give a physical interpretation of this relation, we introduce the *finite* rotation $U_{\mathcal{R}}$ whose relation with U_R is given by (see Fig. 2):

$$U_{\mathcal{R}} = U^{(1)} + U^{(2)} \quad \text{with } U^{(1)} = \theta\wedge OM_0 = U_R \text{ and } U^{(2)} = \theta\wedge(\theta\wedge OM_0). \tag{22}$$

The energy $E_{k_G}(U_R)$ is opposite to the work of the prestress forces (hydrostatic pressure, external forces and weight), considered as "dead loads", during the finite rotation $U_{\mathcal{R}}$ (Morand and Ohayon, 1975).

3.2.3. Computation of $E_{k^2_\Sigma}(U_R)$

We use here the following expression of k^2_Σ:

$$E_{k^2_\Sigma}(U_R) = \frac{1}{2}\int_{\Sigma_0^i} P_0^{\mathrm{app}}\tau\cdot U_R\,\mathrm{d}\Sigma_0. \tag{23}$$

Since for a rotation of vector θ the value of τ is

$$\tau\,\mathrm{d}\Sigma_0 = \theta\wedge n_0\,\mathrm{d}\Sigma_0 + (\theta\cdot n_0)\theta\,\mathrm{d}\Sigma_0, \tag{24}$$

the previous expression gives

$$E_{k^2_\Sigma}(U_R) = \frac{1}{2}\int_{\Sigma_0^i} P_0^{\mathrm{app}}(\theta\wedge n_0)\cdot(\theta\wedge OM_0)\,\mathrm{d}\Sigma_0 + \frac{1}{2}\int_{\Sigma_0^i} P_0^{\mathrm{app}}(\theta\cdot n_0)\underbrace{(\theta\cdot(\theta\wedge OM_0))}_{0}\,\mathrm{d}\Sigma_0. \tag{25}$$

Finally, we find

$$E_{k^2_\Sigma}(U_R) = -\frac{1}{2}\int_{\Sigma_0^i} P_0^{\mathrm{app}} n_0\cdot(\theta\wedge(\theta\wedge OM_0))\,\mathrm{d}\Sigma_0. \tag{26}$$

Let us remark that the deformation energy associated with k^2_Σ exactly compensates for the hydrostatic pressure component of the energy associated with k_G. This result seems in good agreement with physics since the aim of the operator k^2_Σ is to take the follower force effect into account.

3.2.4. Computation of $E_{k_F}(U_R)$

A similar demonstration in this case gives

$$E_{k_F}(U_R) = \frac{1}{2}\int_{\Sigma_0^f} |f_0^F| n_0\cdot(\theta\wedge(\theta\wedge OM_0))\,\mathrm{d}\Sigma_0. \tag{27}$$

3.2.5. Computation of $E_{k_B}(U_R)$

Energy $E_{k_B}(U_R)$ has the following expression:

$$E_{k_B}(U_R) = \frac{\rho^F|g|}{2\|\Gamma_0\|}\left(\underbrace{\int_{\Sigma_0^i}(\theta\wedge OM_0)\cdot n_0\,\mathrm{d}\Sigma_0}_{(A)}\right)^2. \tag{28}$$

Some vector analysis formulae are used to simplify the term denoted as (A). We obtain

$$(A) = \int_{\Omega_0^F}\theta\cdot\underbrace{\mathrm{rot}(OM_0)}_{0}\,\mathrm{d}\Omega_0 + (i_z\wedge\theta)\cdot\left(\int_{\Gamma_0}OM_0\,\mathrm{d}\Gamma_0\right). \tag{29}$$

We introduce C_0, the centre of the liquid free surface (in its initial configuration), defined by $\|\Gamma_0\|OC_0 = \int_{\Gamma_0}OM_0\,\mathrm{d}\Gamma_0$, and finally find for $E_{k_B}(U_R)$:

$$E_{k_B}(U_R) = \frac{\rho^F|g^{\mathrm{app}}|\;\|\Gamma_0\|}{2}((i_z\wedge\theta)\cdot OC_0)^2. \tag{30}$$

3.2.6. Computation of $E_{k_\Sigma^1}(U_R)$

We can write

$$\begin{aligned}E_{k_\Sigma^1}(U_R) &= -\frac{\rho^F|g^{\mathrm{app}}|}{2}\int_{\Sigma_0^i}(OM_0\cdot(i_z\wedge\theta))\,((n_0\wedge\theta)\cdot OM_0)\,\mathrm{d}\Sigma_0\\ &= -\frac{\rho^F|g^{\mathrm{app}}|}{2}\left[\underbrace{\int_{\Omega_0^F}\mathrm{div}((OM_0\cdot(\theta\wedge i_z))(\theta\wedge OM_0))\,\mathrm{d}\Omega_0}_{(A)} + \underbrace{\int_{\Gamma_0}(OM_0\cdot(i_z\wedge\theta))^2\,\mathrm{d}\Gamma_0}_{(B)}\right].\end{aligned} \tag{31}$$

The first term (A) can be simplified to give

$$(A) = (\theta\wedge i_z)\cdot\left(\theta\wedge\int_{\Omega_0^F}OM_0\,\mathrm{d}\Omega_0\right). \tag{32}$$

If the centre of gravity G_0^F of the fluid domain in its initial configuration is introduced, this relation gives

$$(A) = -\|\Omega_0^F\|i_z\cdot(\theta\wedge(\theta\wedge OG_0^F)). \tag{33}$$

The second term (B) is simplified by introducing the point C_0 previously defined. Thus, we obtain

$$(B) = \|\Gamma_0\|(OC_0\cdot(i_z\wedge\theta))^2 + {}^t(i_z\wedge\theta)\mathfrak{I}_\Gamma^C(i_z\wedge\theta), \tag{34}$$

where $\mathfrak{I}_\Gamma^C$ is the inertia tensor of the free surface Γ_0 with respect to its centre C_0, defined by

$$\mathfrak{I}_\Gamma^C = \int_{\Gamma_0}C_0M_0\,{}^t(C_0M_0)\,\mathrm{d}\Gamma_0. \tag{35}$$

Finally, the energy $E_{k_\Sigma^1}(U_R)$ is written as

$$E_{k_\Sigma^1}(U_R) = -\frac{\rho^F|g^{\mathrm{app}}|}{2}(\|\Gamma_0\|(OC_0\cdot(i_z\wedge\theta))^2 + {}^t(i_z\wedge\theta)\mathfrak{I}_\Gamma^C(i_z\wedge\theta)) - \frac{m^F}{2}g^{\mathrm{app}}\cdot(\theta\wedge(\theta\wedge OG_0^F)). \tag{36}$$

3.2.7. Computation of $E_{\mathscr{K}}(U_R)$

The energy $E_{\mathscr{K}}(U_R)$ is obtained by summing all the partial contributions given by Eqs. (18), (21), (26), (27), (30) and (36):

$$E_{\mathscr{K}}(U_R) = -\frac{\rho^F|g^{\mathrm{app}}|}{2}\,{}^t(i_z\wedge\theta)\mathfrak{I}_\Gamma^C(i_z\wedge\theta) - \frac{m}{2}g^{\mathrm{app}}\cdot(\theta\wedge(\theta\wedge OG_0)) - \frac{1}{2}\int_{\Sigma_0^f}f_{D0}\cdot(\theta\wedge(\theta\wedge OM_0))\,\mathrm{d}\Sigma_0, \tag{37}$$

where G_0 is the centre of gravity of the fluid–structure system in its initial configuration and m its mass.

Table 1
Contributions to the deformation energy associated to the rigid body motions

	E_{k_E}	+	$E^{g}_{k_G}$	+	$E^{f_0^F}_{k_G}$	+	E_{k_F}	+	E_{k_B}	+	$E_{k^1_\Sigma}$	+	$E_{k^2_\Sigma}$	=	$E_{\mathscr{K}}$
$T_{x \text{ or } y}$	0	+	0	+	0	+	0	+	0	+	0	+	0	=	0
T_z	0	+	0	+	0	+	0	+	A_1	+	$-A_1$	+	0	=	0
R_z	0	+	A_2	+	A_3	+	$-A_3$	+	0	+	0	+	$-A_2$	=	0
$R_{x \text{ or } y}$	0	+	A_2	+	A_3	+	$-A_3$	+	A_1	+	$A_4 - A_1$	+	$-A_2$	=	A_4

Using the definition of g^{app}, Eq. (2), we can give a final expression of this energy:

$$E_{\mathscr{K}}(U_R) = -\frac{\rho^F|g^{\text{app}}|}{2}\,{}^t(i_z \wedge \theta)\mathfrak{I}^C_\Gamma(i_z \wedge \theta) + \frac{1}{2}\left(\int_{\Sigma_0^f} |f_0^F| n_0 \,\mathrm{d}\Sigma_0\right) \cdot (\theta \wedge (\theta \wedge OG_0)) - \frac{1}{2}\int_{\Sigma_0^f} f_{D0} \cdot (\theta \wedge (\theta \wedge GM_0))\,\mathrm{d}\Sigma_0. \tag{38}$$

The term depending on follower forces f^F is null if f_0^F naturally satisfies the condition of symmetry for k_F ($|f_0^F|$ constant and Σ_0^f closed surface). Otherwise, we can eliminate this term by placing the centre of rotation O on the system centre of gravity G. On the other hand, we remark that the term depending on the dead loads f^D is not cancelled.

We can now give the following conclusions.

(i) If all external loads f_0 are not follower forces, the rotational movements are no longer rigid body modes for the system. We remark that the position of the dead loads in relation to the system centre of gravity determines their stabilizing or destabilizing effect on the system. This result was given in Kreis and Klein (1992).
(ii) If the prestresses in the initial state are only due to follower forces, additional stiffness terms k_Σ and k_F nullify the deformation energy generated by the geometric stiffness k_G. Then, the rotational movement with respect to the gravity axis i_z is still a zero-frequency rigid body mode for the system.
(iii) However, the two rotations with respect to the horizontal axes i_x and i_y are no longer associated with energyless movement, because a fluid–structure system prestressed by the gravity is not invariant during such rotations: when the structure rotates, the liquid free surface remains horizontal by the action of the gravity forces which generate this energy.

Table 1 summarizes the contribution of each term of the stiffness operator to the deformation energy of the system.

4. Numerical examples

4.1. Test case

To illustrate this result, a test case has been studied (Fig. 3 and Table 2). The first free eigenfrequencies, obtained with the hydroelastic modelling with gravity presented here, are shown in Table 3. The study of these results leads to the following remarks.

(i) The x and y-translational modes have zero frequency, as predicted by the theory.
(ii) The translational and rotational modes in relation to the z-axis have quasi-zero frequencies. This error appears because, numerically, the differences between, respectively, E_{k_B} and $E_{k^1_\Sigma}$, and $E^g_{k_G}$ and $E_{k^2_\Sigma}$ are not exactly zero.
(iii) As expected, the rotational motions with respect to x and y axes have nonzero frequencies. These eigenmodes are the result of a coupling between the rotation of the structure and the first fluid sloshing modes (the first sloshing eigenfrequencies are summarized in Table 4). Fig. 4 illustrates the modal participation of the first 48 sloshing modes to the ninth mode (0.59 Hz): it highlights the coupling between the rotational movement of the structure and principally the first and tenth sloshing modes of the fluid.
(iv) The other computed eigenfrequencies are equal to the sloshing frequencies because, in this frequency range, the elastic modes of the structure are not yet excited.

4.2. Comparison with a benchmark

We have tried to validate this aspect of our modelling by comparison with results available in the literature. A benchmark was proposed by Kreis and Klein (1991) to study the coupling between rotational motion of the tank and

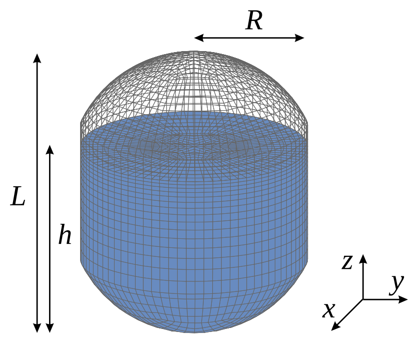

Fig. 3. Mesh of test-case tank.

Table 2
Test case tank definition

R (m)	L (m)	e (mm)	ρ^S (kg/m^3)	E (GPa)	ν	h (m)	ρ^F (kg/m^3)
2.7	6.7	5.0	4450.0	81.4	0.33	4.5	1138.7

Table 3
First free hydroelastic eigenfrequencies with gravity

Mode		Frequency (Hz)
1 and 2	Trans. x and y	0.0
3 and 4	Trans. z and rot. z	$\sim$0.0
5	Rot. x + sloshing 1	0.395
6	Rot. y + sloshing 2	0.395
7 and 8	Sloshing 3 or 4	0.53
9	Rot. x + sloshing 1 et 10	0.59
10	Rot. y + sloshing 2 et 11	0.59
11	Sloshing 5	0.595
12 and 13	Sloshing 6 or 7	0.62

Table 4
First sloshing eigenfrequencies

Mode	1 and 2	3 and 4	5	6 and 7	8 and 9	10 and 11	12 and 13
Frequency (Hz)	0.41	0.53	0.595	0.62	0.70	0.705	0.77

sloshing of the fluid. The proposed model is illustrated by Fig. 5 and Table 5. The tank, whose walls (b) except the bottom (a) are rigid, is suspended from a pivot hinge such that the only rigid body mode of the structure is the rotational motion with respect to the horizontal axis.

Kreis and Klein's method is specially adapted to represent the rigid mode behaviour of the system by including in the model explicit rigid degrees of freedom. Moreover, the fluid is considered through approximations as a subsystem coupled via a pendular motion to the structure. The results we obtain with our modelling on a three-dimensional finite element model are in good agreement with their results: Kreis and Klein predict an eigenmode coupling the rotation of the structure and the first sloshing mode of the fluid for a frequency of 0.99 Hz (see Fig. 6) when we find the same eigenmode with a difference of 8% on the frequency (see Fig. 7). According to the authors' knowledge, no experimental measurements have been compared to the numerical simulation results published on this benchmark, and this will be the topic of future work.

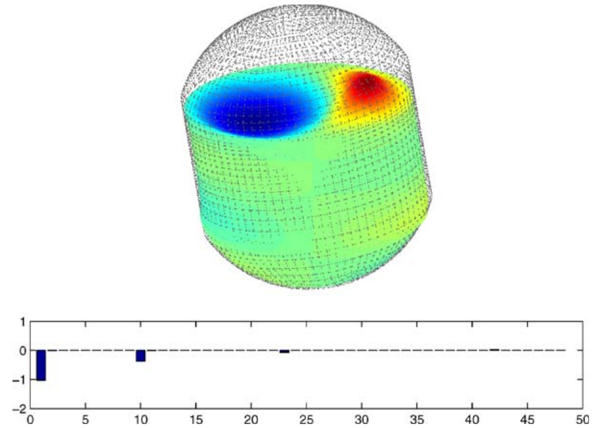

Fig. 4. Sloshing modes participation to ninth free hydroelastic mode.

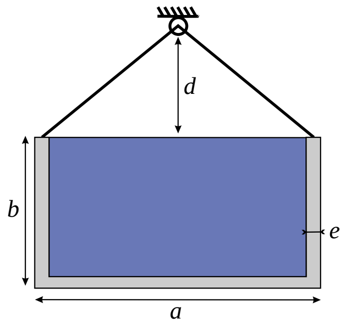

Fig. 5. Kreis and Klein benchmark model.

Table 5
Benchmark model definition

a (m)	b (m)	d (m)	ρ^F (kg/m^3)	e (cm)	ρ^S (kg/m^3)	E (GPa)	ν
1.0	0.5	1.0	1000.0	5.0	3000.0	10.0	0.3

5. Conclusion

We have shown that the hydroelastic modelling with gravity presented in previous papers (Schotté and Ohayon, 1999, 2001) can be adapted to study the particular case of free–free fluid–structure systems by replacing the gravity by an apparent gravity depending on the system acceleration. The study of the rigid body modes has highlighted the prime importance of distinguishing the follower forces from the dead loads in the external forces applied to the system: depending on whether dead loads are applied or not to the system, the rotational motion with respect to the gravity axis can or cannot be a zero-frequency rigid body mode for the system. We have also demonstrated that, due to the presence of gravity, the rotational motions with respect to horizontal axes are not zero-frequency rigid body modes but are coupled with the sloshing movements of the fluid free surface. Experiments are presently undertaken in order to validate the results presented in this paper.

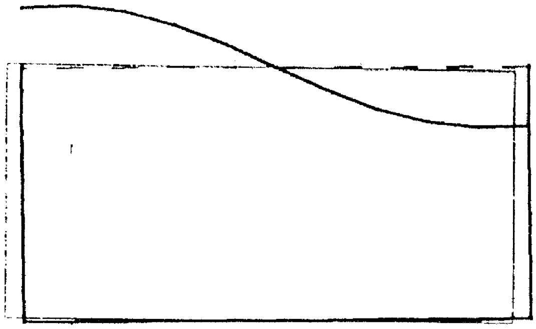

Fig. 6. Eigenmode from Kreis and Klein (1991).

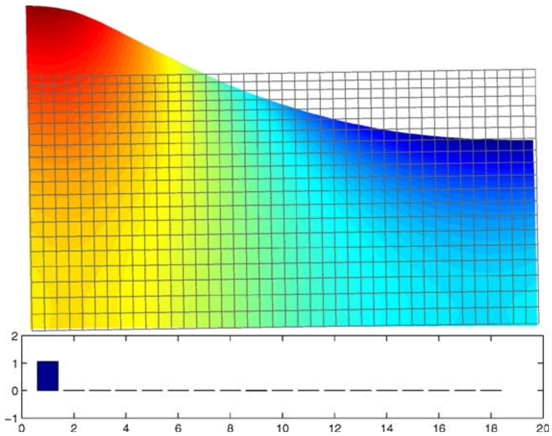

Fig. 7. The same eigenmode as in Fig. 6, computed with our modelling.

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